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A GENERALIZATION OF THE RAYLEIGH DISTRIBUTION

by

Ruth H. Lemon

A thesis submitted in partial fulfillment
of the requirements for the degree

of

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in

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TABLE OF CONTENTS

	Page
1. Summary	1
2. Introduction	3
2.1 Introductory note	3
2.2 Review of some basic concepts used in this paper	4
2.3 Formulas used in this paper	8
2.4 Statement of problem	9
3. A Generalization of the Rayleigh distribution . . .	11
3.1 Derivation of the density function for R . . .	11
3.2 The first and second moments of R	20
3.3 Special cases	22
3.4 The Rayleigh distribution	23
4. Literature cited	25

1. SUMMARY

Let X_1, X_2, \dots, X_n be independent, normal random variables with means m_1, m_2, \dots, m_n respectively, and each having the same variance σ^2 . Let a new random variable R be defined as

$$R = \sqrt{\sum_{i=1}^n X_i^2}$$

The density function for R , and the mean and variance of R are then respectively

$$g(r) = \frac{r^{n-1}}{\sigma^{2(n-2/2)}} \exp \left[-\frac{1}{2\sigma^2} (r^2 + m^2) \right] J_{(n-2/2)} \left(\frac{imr}{\sigma^2} \right) \quad (1)$$

$$\mu = \frac{\sqrt{2}\sigma \Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k}{k! (\frac{n}{2})_k} \left(-\frac{m^2}{2\sigma^2} \right)^k \quad (2)$$

$$\text{Var}(R) = n\sigma^2 + m^2 - 2\sigma^2 \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \left[\sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k}{k! (\frac{n}{2})_k} \left(-\frac{m^2}{2\sigma^2} \right)^k \right]^2 \quad (3)$$

The special cases for $n = 2$ and $n = 3$ are as follows:

$n = 2$

$$g(r) = \left[\frac{r}{\sigma^2} \right] J_0 \left(\frac{imr}{\sigma^2} \right) \exp \left[-\frac{1}{2\sigma^2} (r^2 + m^2) \right] \quad (4)$$

$$\mu = \frac{\sqrt{2\pi}}{2} \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k}{(k!)^2} \left(-\frac{m^2}{2\sigma^2} \right)^k \quad (5)$$

$$\text{Var}(R) = 2\sigma^2 + m^2 - \frac{\pi\sigma^2}{2} \left[\sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k}{(k!)^2} \left(-\frac{m^2}{2\sigma^2}\right)^k \right]^2 \quad (6)$$

$n = 3$

$$g(r) = \sqrt{\frac{2}{\pi}} \frac{r}{m\sigma} \sinh\left(\frac{mr}{\sigma^2}\right) \exp\left[-\frac{1}{2\sigma^2}(r^2 + m^2)\right] \quad (7)$$

$$\mu = 4\sigma\sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} -\frac{1}{k! (4k^2 - 1)} \left(-\frac{m^2}{2\sigma^2}\right)^k \quad (8)$$

$$\text{Var}(R) = 3\sigma^2 + m^2 - \frac{32\sigma^2}{\pi} \left[\sum_{k=0}^{\infty} \frac{1}{k! (4k^2 - 1)} \left(-\frac{m^2}{2\sigma^2}\right)^k \right]^2 \quad (9)$$

If $m = 0$ in equation (1), the density function for R is known as the Raleigh Distribution. In this case it can be shown that (1) reduces to

$$g(r) = \frac{2}{2^n \sigma^n} \frac{r^{n-1}}{\Gamma(\frac{n}{2})} \exp\left[-\frac{r^2}{2\sigma^2}\right] \quad (10)$$

2. INTRODUCTION

2.1 Introductory Note

This paper is divided into numbered sections. The equations are numbered anew in each section, and equation numbers are always enclosed in parentheses. Merely the equation number is given when referring to an equation in the same section as the references; otherwise the section number is prefixed. Thus equation (4) refers to the fourth equation of the same section as the reference, and equation (2.2) refers to the second equation of the second section.

The mathematical notation is that which is in current use, but a few symbols may be explained.

1) The summation sign will be used as \sum , e. g. $\sum_{i=1}^n x_i = x_1 + \dots + x_n$.

2) The ordinary notation for the Gamma-function and the notation of Pochhammer for series of hypergeometric type will be used, i. e.

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt \quad \text{for } x > 0$$

and

$${}_pF_q(a_1, a_2, \dots, a_p; p_1, p_2, \dots, p_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_p)_n}{(p_1)_n (p_2)_n \dots (p_q)_n} z^n$$

where

$$(a)_n = a(a-1)(a-2)\dots(a-n+1), \quad (a)_0 = 1.$$

In particular

$${}_1F_1(a;p;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n! (p)_n} z^n .$$

3) Where the exponent is concise, the exponential function will be written as a power for e , for example e^x . But where it is lengthy we shall use the notation exemplified by $\exp\left[-\frac{1}{2}(x^2 - 2rxy - y^2)\right]$ instead of $e^{-\frac{1}{2}(x^2 - 2rxy - y^2)}$.

2.2 Review of Some Basic Concepts Used in This Paper

Continuous random variables.-- A continuous random variable, X , has associated with it a probability density function, $f(x)$, which has the following properties.

1) $f(x)$ is a real valued function defined over the set of real numbers.

2) $f(x) \geq 0$ for all $-\infty < x < \infty$.

3) $\int_{-\infty}^{\infty} f(x) dx = 1$.

The probability that X belongs to a set A is defined as the integral of $f(x)$ over the set A . In particular, if A is the interval $a < x < b$, then

$$P[a < X < b] = \int_a^b f(x) dx .$$

It becomes convenient to consider a distribution function, $F(x)$, which is defined as

$$F(x) = \int_{-\infty}^x f(t) dt.$$

Then we can write

$$P[a < X < b] = F(b) - F(a) .$$

The above defined distribution function has the properties of being non-decreasing, $\lim_{x \rightarrow -\infty} F(x) = 0$, and $\lim_{x \rightarrow \infty} F(x) = 1$.

If the distribution function, $F(x)$, is given, the density function can be determined by the relationship

$$f(x) = \frac{d}{dx} F(x) .$$

Expectation of a random variable and moments.---The expectation $E(X)$ of X is defined by the following integral (whenever the integral exists).

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx .$$

In general, if g is any function, then the expectation of the new random variable $g(X)$ is defined as

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x) dx.$$

In particular, $E(X^r)$, where r is a non-negative integer, is called the r^{th} moment of X about the origin and is denoted by μ_r' . That is,

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x) dx .$$

The first moment is used extensively and is called the mean of X , which is denoted by μ .

The r^{th} moment about μ is defined as $E[(X - \mu)^r]$ and is denoted by μ_r .

$$\mu_r = \int_{-\infty}^{\infty} (x - \mu)^r f(x) dx .$$

The second moment about the mean, μ_2 , is called the variance of X and is represented by σ^2 . The positive square root is called the standard deviation of X .

Joint density functions.---The above notion of a density function can be extended to density functions of two or more random variables. For example, $f(x,y)$ is a joint density function of two random variables X and Y if:

- 1) the domain of $f(x,y)$ is the xy -plane.
- 2) $f(x,y) \geq 0$ for all $-\infty < x < \infty$, $-\infty < y < \infty$.
- 3) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dx dy = 1$.

The density functions, $f_1(x)$ and $f_2(y)$, associated with X and Y can be obtained as follows.

$$f_1(x) = \int_{-\infty}^{\infty} f(x,y) dy$$

$$f_2(y) = \int_{-\infty}^{\infty} f(x,y) dx$$

(These are called marginal density functions). The probability that point (X,Y) is in region A of the xy -plane is defined as the double integral of $f(x,y)$ over region A .

Random variables X and Y are called independent (in the probability sense) if

$$f(x,y) = f_1(x) f_2(y) .$$

The above concepts can be extended to joint density functions for n random variables.

If X_1, X_2, \dots, X_n are random variables having joint density function $f(x_1, x_2, \dots, x_n)$, and if Z is a random variable defined by $Z = h(X_1, X_2, \dots, X_n)$, then the distribution function for Z is given by

$$F(z) = \int \dots \int f(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n \dots \quad (1)$$

where the integration is over the domain defined by $h(x_1, x_2, \dots, x_n) < z$. This method of expressing $F(z)$ will be used in this paper. The density function associated with Z will be found by differentiating $F(z)$ with respect to z .

The Normal Distribution.--The density function for the normal random variable with mean μ and variance σ^2 is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left[-\frac{1}{2\sigma^2} (x - \mu)^2 \right], \quad -\infty < x < \infty .$$

The distribution function is given by

$$F(x) = \left(\frac{1}{\sqrt{2\pi}} \right) \int_{-\infty}^{(x-\mu)/\sigma} e^{-\frac{1}{2}t^2} dt .$$

$$\int_0^{\infty} J_v(at) e^{-p^2 t^2} t^{v+1} dt = \frac{a^v}{(2p^2)^{v+1}} \exp\left[-\frac{a^2}{4p^2}\right], \quad \dots \quad (5)$$

provided $R(v) > -1$.

2.4 Statement of Problem

The problem which is considered in this paper may be stated as follows: Let X_1, X_2, \dots, X_n be independent, normal random variables with means m_1, m_2, \dots, m_n respectively, and each having the same variance σ^2 . Let a new random variable, R , be defined by

$$R = \sqrt{\sum_{i=1}^n x_i^2} \quad \dots \quad (6)$$

The problem is to find the density function for R , and the mean and variance of R .

If $m_i = 0$ for $i = 1, 2, \dots, n$, the density function for R , $g(r)$, is known as the Rayleigh Distribution (1, p. 236) and can easily be shown to be

$$g(r) = \frac{2}{2^{n/2} \sigma^n \Gamma(\frac{n}{2})} r^{n-1} e^{-r^2/2\sigma^2} \quad \dots \quad (7)$$

Procedure.--The density function for the general case of n variables will be considered first; and the density function for R will be called A Generalization of the Rayleigh Distribution. The results for $n = 2$ and $n = 3$ will be stated explicitly since these may be of practical significance. The mean and variance of R for the general case will be determined, and then explicitly given for $n = 2$

and $n = 3$. Finally, as a check, it will be shown that $m_i = 0$ for $i = 1, 2, \dots, n$, the Generalization of the Rayleigh Distribution agrees with (7).

3. A GENERALIZATION OF THE RAYLEIGH DISTRIBUTION

3.1 Derivation of the Density Function for R

From the statement of the problem, it is assumed that there exists a set of random variables $[X_i]$, $i = 1, 2, \dots, n$, which are normally distributed and independent. Therefore the joint density function for these variables will be

$$f(x_1, x_2, \dots, x_n) = \frac{1}{(2\pi)^{n/2} \sigma^n} \exp \left[-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - m_i)^2 \right]. \quad (1)$$

The new random variable R is defined by

$$R = \sqrt{\sum_{i=1}^n x_i^2}.$$

If we let

$$G(u) = P[R \leq u] = \int \dots \int f(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n, \quad (2)$$

where the integration is taken over the domain of the X 's such that $R \leq u$, the density function which we are seeking will be determined by

$$g(u) = \frac{d}{du} G(u).$$

By substituting the right side of (1) into the right side of (2), it can be seen that

$$G(u) = \frac{1}{(2\pi)^{n/2} \sigma^n} \int \dots \int \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - m_i)^2 \right\} dx_1 dx_2 \dots dx_n. \quad (3)$$

It is now convenient to consider the two vectors, $\bar{r} = (x_1, x_2, \dots, x_n)$ and $\bar{m} = (m_1, m_2, \dots, m_n)$. It is apparent that the numerator of the exponent in the above expression becomes the vector dot product $(\bar{r} - \bar{m}) \cdot (\bar{r} - \bar{m})$. Therefore (3) becomes

$$G(u) = \frac{1}{(2\pi)^{n/2} \sigma^n} \int \dots \int \exp \left[-\frac{1}{2\sigma^2} (r^2 + m^2 - 2rm \cos \alpha) \right] dx_1 dx_2 \dots dx_n \quad (4)$$

where $r = |\bar{r}|$, $m = |\bar{m}|$, and α is the angle between \bar{r} and \bar{m} .

In order to simplify the integration, the following transformation is made.

$$\begin{aligned} x_1 &= r \sin \theta_1 \sin \theta_2 \dots \sin \theta_{n-1} \\ x_2 &= r \sin \theta_1 \sin \theta_2 \dots \sin \theta_{n-2} \cos \theta_{n-1} \\ &\dots \\ x_i &= r \sin \theta_1 \sin \theta_2 \dots \sin \theta_{n-i} \cos \theta_{n-i+1} \\ &\dots \\ x_n &= r \cos \theta_1 \end{aligned}$$

The Jacobian of this transformation is given by

$$J = \frac{\partial(x_1, x_2, \dots, x_n)}{\partial(r, \theta_1, \dots, \theta_{n-1})}.$$

It can be seen that the Jacobian is then equal to r^{n-1} times the following determinant.

$$\Delta = \begin{vmatrix} \sin \theta_1 \sin \theta_2 \dots \sin \theta_{n-1} & \sin \theta_1 \sin \theta_2 \dots \sin \theta_{n-2} \cos \theta_{n-1} & \dots & \cos \theta_1 \\ \cos \theta_1 \sin \theta_2 \dots \sin \theta_{n-1} & \cos \theta_1 \sin \theta_2 \dots \sin \theta_{n-2} \cos \theta_{n-1} & \dots & -\sin \theta_1 \\ \sin \theta_1 \cos \theta_2 \dots \sin \theta_{n-1} & \sin \theta_1 \cos \theta_2 \dots \sin \theta_{n-2} \cos \theta_{n-1} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \sin \theta_1 \sin \theta_2 \dots \cos \theta_{n-1} & -\sin \theta_1 \sin \theta_2 \dots \sin \theta_{n-1} & \dots & 0 \end{vmatrix}$$

Taking out common factors in columns, Δ reduces to $\sin^{n-1} \theta_1 \sin^{n-2} \theta_2 \dots \sin \theta_{n-1} \cos \theta_1 \cos \theta_2 \dots \cos \theta_{n-1}$ times

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ \cot \theta_1 & \cot \theta_1 & \cot \theta_1 & \dots & -\tan \theta_1 \\ \cot \theta_2 & \cot \theta_2 & \cot \theta_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \cot \theta_{n-2} & \cot \theta_{n-2} & -\tan \theta_{n-2} & \dots & 0 \\ \cot \theta_{n-1} & -\tan \theta_{n-1} & 0 & \dots & 0 \end{vmatrix}$$

If this determinant, by subtracting each column from the preceding one, Δ is found to reduce to $\sin^{n-2} \theta_1 \sin^{n-3} \theta_2 \dots \sin \theta_{n-2}$, thus

$$J = r^{n-1} \sin^{n-2} \theta_1 \dots \sin \theta_{n-2}.$$

The advantage of using this transformation is that the limits of the integral in (4) are now much simpler. r itself can vary from zero to u , θ_{n-1} can vary from 0 to 2π and the other θ 's vary from 0 to π .

If the axes are rotated until the vector \bar{m} lies along the positive x_2 axis, it can be shown that $\cos \alpha$ becomes

$$\cos \alpha = \sin \theta_1 \sin \theta_2 \dots \cos \theta_{n-1}.$$

The Jacobian of this rotation is 1.

Thus (4) becomes

$$G(u) = \frac{1}{(2\pi)^{n/2} \sigma^n} \int_0^\pi \dots \int_0^\pi \int_0^{2\pi} \int_0^u \exp \left\{ -\frac{1}{2\sigma^2} (r^2 + m^2 - 2rm \sin \theta_1 \dots \cos \theta_{n-1}) \right\} r^{n-1} \sin^{n-2} \theta_1 \sin^{n-3} \theta_2 \dots \sin \theta_{n-2} dr d\theta_{n-1} \dots d\theta_1 \quad (5)$$

The derivative of $G(u)$ with respect to u becomes

$$\frac{d}{du} G(u) = \frac{u^{n-1}}{(2\pi)^{n/2} \sigma^n} \exp \left\{ -\frac{1}{(2\sigma^2)} (u^2 + m^2) \right\} \int_0^\pi \dots \int_0^\pi \int_0^{2\pi} \exp \left\{ -\frac{um}{\sigma^2} \sin \theta_1 \dots \cos \theta_{n-1} \right\} \sin^{n-2} \theta_1 \dots \sin \theta_{n-2} d\theta_{n-1} \dots d\theta_1 \quad (6)$$

Formally (6) is the solution to our problem. The problem now reduces to one of evaluating the multiple integral appearing in (6).

In order to evaluate the multiple integral, it becomes desirable to consider the integral

$$\int_0^\pi \dots \int_0^\pi \exp (iu \sin \theta_1 \dots \cos \theta_{n-1}) \sin^{n-2} \theta_1 \dots \sin \theta_{n-2} d\theta_{n-1} \dots d\theta_1.$$

It is possible to obtain a new integral in quite a natural manner by means of transformation of a type used in the geometry of the sphere.

(4, p. 51) Consider θ_{n-1} and θ_{n-2} as longitude and colatitude of a point on a unit sphere; and denote the direction-cosines of the vector

from the center to this point by (h, k, m) , and the element of surface at this point by dA . The integral can then be transformed by making a cyclical interchange of the co-ordinate axes in the following manner:¹

If h, k, m , are the direction-cosines then,

$$h = \sin \theta_{n-2} \cos \theta_{n-1}$$

$$k = \sin \theta_{n-2} \sin \theta_{n-1}$$

$$m = \cos \theta_{n-2}$$

$$\begin{aligned} & \int_0^\pi \dots \int_0^\pi \int_0^\pi \exp(iw \sin \theta_1 \dots \sin \theta_{n-2} \cos \theta_{n-1}) \sin^{n-2} \theta_1 \dots \sin \theta_{n-2} d\theta_{n-1} \dots d\theta_1 \\ &= \int \dots \iint_S \exp(iw \sin \theta_1 \dots \sin \theta_{n-3} h) \sin^{n-2} \theta_1 \dots \sin^2 \theta_{n-3} dA d\theta_{n-3} \dots d\theta_1 \\ &= \int \dots \iint_S \exp(iw \sin \theta_1 \dots \sin \theta_{n-3} m) \sin^{n-2} \theta_1 \dots \sin^2 \theta_{n-3} dA d\theta_{n-3} \dots d\theta_1 \\ &= \int_0^\pi \dots \int_0^\pi \int_0^\pi \exp(iw \sin \theta_1 \dots \sin \theta_{n-3} \cos \theta_{n-2}) \sin^{n-2} \theta_1 \dots \sin \theta_{n-2} d\theta_{n-1} \dots d\theta_1 . \end{aligned}$$

Integrating with respect to θ_{n-1} , we get a constant k_1 times

$$\int_0^\pi \dots \int_0^\pi \exp(iw \sin \theta_1 \dots \sin \theta_{n-3} \cos \theta_{n-2}) \sin^{n-2} \theta_1 \dots \sin \theta_{n-2} d\theta_{n-2} \dots d\theta_1 .$$

¹The symbol \iint_S means that the integration extends over the

the surface of the hemisphere on which m is positive.

Again, by considering θ_{n-2} and θ_{n-3} as longitude and colatitude of a point on a unit sphere; and again denoting the direction-cosines of the vector from the center to this point by (h, k, m) , and the element of surface at this point by dA , then

$$h = \sin \theta_{n-3} \cos \theta_{n-2}$$

$$k = \sin \theta_{n-3} \sin \theta_{n-2}$$

$$m = \cos \theta_{n-3}$$

$$\int_0^\pi \dots \int_0^\pi \int_0^\pi \exp(iw \sin \theta_1 \dots \sin \theta_{n-3} \cos \theta_{n-2}) \sin^{n-2} \theta_1 \dots \sin \theta_{n-2} d\theta_{n-2} \dots d\theta_1$$

$$= \int \dots \iint_{S_{m \neq 0}} \exp(iw \sin \theta_1 \dots \sin \theta_{n-4} h) \sin^{n-2} \theta_1 \dots k dA d\theta_{n-4} \dots d\theta_1$$

$$= \int \dots \iint_{S_{k \neq 0}} \exp(iw \sin \theta_1 \dots \sin \theta_{n-4} m) \sin^{n-2} \theta_1 \dots h dA d\theta_{n-4} \dots d\theta_1$$

$$= \int_0^\pi \dots \int_0^\pi \int_0^\pi \exp(iw \sin \theta_1 \dots \sin \theta_{n-4} \cos \theta_{n-3}) \sin^{n-2} \theta_1 \dots \sin^2 \theta_{n-3} \dots d\theta_{n-2} \dots d\theta_1$$

Integrating with respect to θ_{n-2} , we get a constant k_2 times

$$\int_0^\pi \dots \int_0^\pi \int_0^\pi \exp(iw \sin \theta_1 \dots \sin \theta_{n-4} \cos \theta_{n-3}) \sin^{n-2} \theta_1 \dots \sin^2 \theta_{n-3} d\theta_{n-3} \dots d\theta_1$$

Again repeating this process, this integral can be reduced in the same manner.

By repeating the above process $n-2$ times it can easily be seen that

$$\int_0^\pi \dots \int_0^\pi \int_0^\pi \exp(iw \sin \theta_1 \dots \sin \theta_{n-2} \cos \theta_{n-1}) \sin^{n-2} \theta_1 \dots \sin \theta_{n-2} d\theta_{n-1} \dots d\theta_1$$

$$= K \int_0^\pi \exp(iw \cos \theta_1) \sin^{n-2} \theta_1 d\theta_1 \dots \dots \dots (7)$$

It is now convenient to consider the expression

$$y = w^v \int_0^\pi \exp(iw \cos \theta) \sin^{2v} \theta d\theta \dots \dots \dots (8)$$

where v is a constant and show it now satisfies Bessel's differential equation

$$w^2 \frac{d^2 y}{dw^2} + w \frac{dy}{dw} + (w^2 - v^2)y = 0 \dots \dots \dots (9)$$

It can be shown that

$$w^v \int_0^\pi \exp(iw \cos \theta) \sin^{2v} \theta d\theta = w^v \int_0^\pi \cos(w \cos \theta) \sin^{2v} \theta d\theta \dots$$

The first and second derivatives of y with respect to w are then

$$\frac{dy}{dw} = \int_0^\pi \left[-w^v \sin^{2v} \theta \cos \theta \sin(w \cos \theta) + v w^{v-1} \sin^{2v} \theta \cos(w \cos \theta) \right] d\theta \quad (10)$$

$$\frac{d^2 y}{dw^2} = \int_0^\pi \left[-w^v \sin^{2v} \theta \cos^2 \theta \cos(w \cos \theta) - 2vw^{v-1} \sin^{2v} \theta \cos \theta \sin(w \sin \theta) \right. \\ \left. + v(v-1)w^{v-2} \cos(w \cos \theta) \sin^{2v} \theta \right] d\theta \quad \dots \quad (11)$$

Substituting (10) and (11) into the left side of (9) gives

$$w^{v+2} \int_0^\pi \sin^{2v} \theta \sin^2 \theta \cos(w \cos \theta) d\theta \\ - (2v+1)w^{v+1} \int_0^\pi \sin^{2v} \theta \cos \theta \sin(w \cos \theta) d\theta \quad (12)$$

Integrating the second integral in the above expression by parts, (12) becomes

$$w^{v+2} \int_0^\pi \sin^{2v+2} \theta \cos(w \cos \theta) d\theta - w^{v+2} \int_0^\pi \sin^{2v+2} \theta \cos(w \cos \theta) d\theta = 0.$$

It can therefore be concluded that (8) is a solution of Bessel's differential equation.

The complete solution of Bessel's differential equation is

$$y = C_1 J_v(w) + C_2 Y_v(w) \quad \dots \quad (13)$$

where J_v and Y_v are Bessel functions of the first and second kind respectively of order v , and C_1 and C_2 are arbitrary constants.

It has been shown that

$$y = w^v \int_0^\pi \exp(iw \cos \theta) \sin^{2v} \theta d\theta$$

is a solution of Bessel's differential equation. It can be seen that this expression is always finite for all finite values of w .

Since $Y_v(w)$ becomes infinite when $w = 0$, the constant C_2 in (13) will be taken as zero, and we have

$$C_1 J_v(w) = w^v \int_0^\pi \exp(iw \cos \theta) \sin^{2v} \theta d\theta. \quad (14)$$

By letting $2v = n-2$ in equation (14), and making use of the results in equation (7), it can be seen that

$$\begin{aligned} \int_0^\pi \dots \int_0^\pi \exp(iw \sin \theta_1 \dots \sin \theta_{n-2} \cos \theta_{n-1}) \sin^{n-2} \theta_1 \dots \sin \theta_{n-2} d\theta_{n-1} \dots d\theta_1 \\ = \frac{C J_{(n-2/2)}(w)}{w^{(n-2/2)}} \end{aligned} \quad (15)$$

These results will now be used to evaluate the multiple integral appearing in (6). Equation (6) states

$$\begin{aligned} g(r) = \frac{r^{n-1}}{(2\pi)^{n/2} \sigma^n} \exp \left[-\frac{1}{2\sigma^2} (r^2 + m^2) \right] \int_0^\pi \dots \int_0^{2\pi} \exp \left[-\frac{rm}{\sigma^2} \sin \theta_1 \dots \right. \\ \left. \sin \theta_{n-2} \cos \theta_{n-1} \right] \sin^{n-2} \theta_1 \dots \sin \theta_{n-2} d\theta_{n-1} \dots d\theta_1. \end{aligned} \quad (16)$$

If in (15) we let $w = \frac{imr}{\sigma^2}$, it can be seen that (16) becomes

$$g(r) = \frac{C r^{n-1}}{(2\pi)^{n/2} \sigma^n (imr)^{(n-2/2)}} J_{(n-2/2)} \left(\frac{imr}{\sigma^2} \right) \exp \left[-\frac{1}{2\sigma^2} (r^2 + m^2) \right]. \quad (17)$$

The constant C may be evaluated by integrating $g(r)$ from 0 to ∞ as follows.

$$1 = \frac{C \exp\left[-\frac{m^2}{2\sigma^2}\right]}{(2\pi)^{n/2} \sigma^n (im)^{(n-2/2)}} \int_0^\infty J_{(n-2/2)}\left(\frac{imr}{\sigma^2}\right) \exp\left[-\frac{r^2}{2\sigma^2}\right] r^{n/2} dr. \quad (18)$$

By making use of the results from equation (2.5), (18) becomes

$$\begin{aligned} 1 &= \frac{C \exp\left[-\frac{m^2}{2\sigma^2}\right] \left(\frac{im}{\sigma^2}\right)^{(n-2/2)} \sigma^n}{(2\pi)^{n/2} \sigma^n (im)^{(n-2/2)}} \exp\left[\frac{m^2}{2\sigma^2}\right] \\ &= \frac{C}{\sigma^{n-2} (2\pi)^{n/2}}. \end{aligned}$$

Therefore

$$C = \sigma^{n-2} (2\pi)^{n/2}.$$

The Generalization of the Raleigh Distribution is therefore given by

$$g(r) = \frac{r^{n-1}}{\sigma^2 (imr)^{(n-2/2)}} \exp\left[-\frac{1}{2\sigma^2} (r^2 + m^2)\right] J_{(n-2/2)}\left(\frac{imr}{\sigma^2}\right). \quad (19)$$

3.2 The First and Second Moments of R

The first moment about the origin, or mean of R will be

$$\mu_1' = E(R) = \int_0^\infty r g(r) dr$$

$$= \frac{\exp \left[-\frac{m^2}{2\sigma^2} \right]}{(im)^{(n-2/2)} \sigma^2} \int_0^\pi r^{(n+2/2)} \exp \left[-\frac{r^2}{2\sigma^2} \right] J_{(n-2/2)} \left(\frac{imr}{\sigma^2} \right) dr$$

By applying equation (2.4), this expression becomes

$$\begin{aligned} \mu'_1 &= \frac{\exp \left[-\frac{m^2}{2\sigma^2} \right] \Gamma \left(\frac{n+1}{2} \right) \left(\frac{\sqrt{2} im}{2\sigma^2} \right)^{(n-2/2)}}{(im)^{(n-2/2)} \sigma^2 \Gamma \left(\frac{n}{2} \right) \left(\frac{1}{\sqrt{2}\sigma} \right)^{(n+4/2)}} \exp \left(\frac{m^2}{2\sigma^2} \right) {}_1F_1 \left(-\frac{1}{2}; \frac{n}{2}; -\frac{m^2}{2\sigma^2} \right) \\ &= \frac{\sqrt{2}\sigma \Gamma \left(\frac{n+1}{2} \right)}{\Gamma \left(\frac{n}{2} \right)} \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{2} \right)_k}{k! \left(\frac{n}{2} \right)_k} \left(-\frac{m^2}{2\sigma^2} \right)^k \end{aligned}$$

Therefore the mean of R is

$$\mu = \frac{\sqrt{2}\sigma \Gamma \left(\frac{n+1}{2} \right)}{\Gamma \left(\frac{n}{2} \right)} \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{2} \right)_k}{k! \left(\frac{n}{2} \right)_k} \left(-\frac{m^2}{2\sigma^2} \right)^k \quad (20)$$

The second moment about the origin of R will be

$$\begin{aligned} \mu'_2 = E(R^2) &= \int_0^\infty r^2 g(r) dr = \frac{\exp \left[-\frac{m^2}{2\sigma^2} \right]}{(im)^{(n-2/2)} \sigma^2} \\ &\cdot \int_0^\infty J_{(n-2/2)} \left(\frac{imr}{\sigma^2} \right) \exp \left[-\frac{r^2}{2\sigma^2} \right] r^{(n+4/2)} dr. \end{aligned}$$

From the results of equation (2.2),

$$\mu'_2 = \frac{\exp \left[-\frac{m^2}{2\sigma^2} \right] \Gamma \left(\frac{n+2}{2} \right) (\sqrt{2}\sigma)^{(n+6/2)} \left(\frac{im}{\sqrt{2}\sigma} \right)^{(n-2/2)}}{2 (im)^{(n-2/2)} \sigma^2 \Gamma \left(\frac{n}{2} \right)} {}_1F_1 \left(\frac{n+2}{2}; \frac{n}{2}; \frac{m^2}{2\sigma^2} \right)$$

$$\text{Var}(r) = 2\sigma^2 + m^2 - \frac{\pi}{2} \sigma^2 \left[\sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k}{(k!)^2} \left(-\frac{m^2}{2\sigma^2}\right)^k \right]^2 \quad (26)$$

$$\underline{n = 3}$$

$$g(r) = \frac{r^2}{\sigma^2 (imr)^{\frac{1}{2}}} J_{\frac{1}{2}} \left(\frac{imr}{\sigma^2} \right) \exp \left[-\frac{1}{2\sigma^2} (r^2 + m^2) \right] \quad (27)$$

by making use of the well-known identity

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

equation (27) reduces to

$$g(r) = \sqrt{\frac{2}{\pi}} \frac{r}{m\sigma} \sinh \left(\frac{mr}{\sigma^2} \right) \exp \left[-\frac{1}{2\sigma^2} (r^2 + m^2) \right] \quad (28)$$

$$\mu = 4\sigma \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} -\frac{1}{k! (4k^2-1)} \left(-\frac{m^2}{2\sigma^2}\right)^k \quad (29)$$

$$\mu'_2 = 3\sigma^2 + m^2 \quad (30)$$

$$\text{Var}(R) = 3\sigma^2 + m^2 - \frac{32\sigma^2}{\pi} \left[\sum_{k=0}^{\infty} -\frac{1}{k! (4k^2-1)} \left(-\frac{m^2}{2\sigma^2}\right)^k \right]^2 \quad (31)$$

3.4 The Rayleigh Distribution

It will now be verified that when $m = 0$, (19) becomes

$$g(r) = \frac{2}{2^{n/2} \sigma^n \Gamma(\frac{n}{2})} r^{n-1} \exp \left[-\frac{r^2}{2\sigma^2} \right] \quad (32)$$

Equation (19) states

$$\begin{aligned}
 g(r) &= \frac{r^{n-1}}{(imr)^{(n-2/2)} \sigma^2} J_{(n-2/2)} \left(\frac{imr}{\sigma^2} \right) \exp \left[-\frac{1}{2\sigma^2} (r^2 + m^2) \right] \\
 &= \frac{r^{n-1}}{\sigma^2} \exp \left[-\frac{1}{2\sigma^2} (r^2 + m^2) \right] \frac{1}{(imr)^{(n-2/2)}} \\
 &\quad \cdot \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{imr}{\sigma^2} \right)^{(2k + \frac{n}{2} - 1)}}{2^{(2k + \frac{n}{2} - 1)} k! (k + \frac{n}{2} - 1)!} \\
 &= \frac{2r^{n-1}}{2^{n/2} \sigma^n} \exp \left[-\frac{1}{2\sigma^2} (r^2 + m^2) \right] \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{imr}{\sigma^2} \right)^{2k}}{2^{2k} k! (k + \frac{n}{2} - 1)!} \\
 &= \frac{2r^{n-1}}{2^{n/2} \sigma^n} \exp \left[-\frac{1}{2\sigma^2} (r^2 + m^2) \right] \left[\frac{1}{(\frac{n}{2} - 1)!} - \frac{(\frac{imr}{\sigma^2})^2}{2^2 (\frac{n}{2})!} \right. \\
 &\quad \left. + \frac{(\frac{imr}{\sigma^2})^4}{2^4 2! (\frac{n}{2} - 1)!} - \dots \right]
 \end{aligned}$$

Now by substituting $m = 0$ into the above expression, it can easily be seen that $g(r)$ becomes

$$g(r) = \frac{2r^{n-1}}{2^{n/2} \sigma^n \Gamma(\frac{n}{2})} \exp \left[-\frac{r^2}{2\sigma^2} \right]$$

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